JOURNAL OF GUIDANCE, CONTROL, AND DYNAMICS
Vol. 20, No. 3, May–June 1997

Higher-Order Cayley Transforms with Applications to Attitude Representations

Panagiotis Tsiotras*
University of Virginia, Charlottesville, Virginia 22903-2442
and
John L. Junkins† and Hanspeter Schaub‡
Texas A&M University, College Station, Texas 77843-3141

We generalize previous results on attitude representations by using Cayley transforms. First, we show that proper orthogonal matrices, which naturally represent rotations, can be generated by a form of conformal analytic mappings in the space of matrices. Using a natural parallelism between the elements of the complex plane and the real matrices, we generate higher-order Cayley transforms and discuss some of their properties. These higher-order Cayley transforms are shown to parameterize proper orthogonal matrices into higher-order Rodrigues parameters.

I. Introduction

THE question of the proper choice of coordinates for describing rotations has a very long and exciting history. Starting with the work of Euler and Hamilton, a series of different parameterizations have been introduced by several researchers during the past hundred years. We will not delve into these results here because they can be found in any good textbook on attitude representations.[1,2] We just mention the work of Stuelpnagel in this area,[3] as well as the recent survey article by Shuster[4] in the special issue of the *Journal of the Astronautical Sciences* on Attitude Representations.

In this paper, we take a point of view slightly more abstract than the previous references. Our main objective is to unify some of the existing results in the area of attitude representations. It is hoped that this global view will add to current understanding of attitude representations. Our motivation stems mainly from the recent results on second-order Rodrigues parameters.[5–7] In particular, in Ref. 7 it was shown that these (modified) Rodrigues parameters can be generated by a second-order Cayley transform the same way the classic Cayley–Rodrigues parameters are generated by the Cayley transform.[1,8] Viewing the Cayley transform as a bilinear transformation that maps the space of skew-symmetric matrices onto the space of proper orthogonal matrices (and vice versa), one is naturally led to the notion of conformal mappings (a generalization of the bilinear transformation) from the imaginary axis onto the unit circle (and vice versa). We seek to generalize these conformal mappings to matrix spaces. Drawing on the insightful statements by Halmos,[9] we show that such an intuitive generalization is indeed possible. We are therefore able to generate the Euler parameters, the Rodrigues parameters, and the modified Rodrigues parameters as special cases of such conformal mappings. Higher-order Rodrigues parameters can be easily constructed by this approach, although their relevance to applications is still to be determined. We explicitly develop the third- and fourth-order Rodrigues parameters to illustrate potential advantages as well as difficulties. The question of kinematics of these higher order Rodrigues parameters is briefly discussed in the last section of the paper. A more in-depth discussion of the kinematics is left for future investigation. The presentation and derivations of the results are kept as formal as possible.

The first part of the paper reviews the standard Cayley transform, and it generalizes this transform to higher orders. There is no restriction on the dimension of the matrices involved, i.e., the results hold for $n \times n$ matrices. In the second part of the paper, we apply these results to the case of interest to attitude dynamicists, i.e., the case $n = 3$.

Some notation and terminology are necessary to keep the discussion clear and terse. We use the standard mathematical notation $SO(n)$ to denote the space of proper orthogonal matrices of dimension $n \times n$. The space of orthogonal matrices is denoted by $O(n)$ and it is the set of all (invertible) matrices such that $A^TA = AA^T = I$. Clearly, if $A \in O(n)$ then $\det(A) = \pm 1$. The qualifier proper then refers to those orthogonal matrices with a positive determinant, that is, $SO(n) = \{A \in \mathbb{R}^{n\times n} : AA^T = I, \det(A) = +1\}$. These matrices represent rotations, whereas the orthogonal matrices with determinant -1 involve, in general, reflections.[10] The space $SO(n)$ [as well as $O(n)$] forms a group. We will see later on that one can define a differential equation for elements of $SO(n)$. The solutions of this differential equation form trajectories (one-parameter subgroups) on $SO(n)$ and this differentiable structure makes $SO(n)$ actually a Lie group (i.e., a group with a differentiable manifold structure). The space of $n \times n$ skew-symmetric matrices is denoted by $so(n)$. That is, $so(n) = \{A \in \mathbb{R}^{n\times n} : A = -A^T\}$. The space $so(n)$ is actually the tangent vector space to $SO(n)$ at the identity.[10] Finally, following standard mathematical language, we use the symbols $\mathbb{C}$, $\mathbb{R}$, $\Im = i\mathbb{R}$, and S^1 to denote the complex numbers, the real numbers, the imaginary numbers, and the numbers with absolute value one (i.e., the numbers on the unit circle), respectively. The symbol $sp(\cdot)$ denotes the spectrum of a matrix, i.e., the set of its eigenvalues.

II. Cayley Transform

Cayley's transformation parameterizes a proper orthogonal matrix, C, as a function of a skew-symmetric matrix, Q. It is, therefore, a map $\psi : so(n) \to SO(n)$. The classic Cayley transform[8] is given by

$$C = \psi(Q) = (I - Q)(I + Q)^{-1} = (I + Q)^{-1}(I - Q) \qquad (1)$$

Because Q is skew-symmetric, all of its eigenvalues are pure imaginary. Thus, all the eigenvalues of the matrix $I + Q$ are nonzero and the inverse in Eq. (1) exists. The Cayley transform is therefore well-defined for all skew-symmetric matrices. The inverse transformation is identical and is given by

$$Q = \psi^{-1}(C) = \psi(C) = (I - C)(I + C)^{-1}$$
$$= (I + C)^{-1}(I - C) \qquad (2)$$

The inverse transformation is not defined when C has an eigenvalue at -1, because in this case $\det(I + C) = 0$. Because C is

Presented as Paper 96-3628 at the AIAA/AAS Astrodynamics Specialist Conference, San Diego, CA, July 29–31, 1996; received Sept. 2, 1996; revision received Feb. 14, 1997; accepted for publication Feb. 16, 1997.

*Assistant Professor, Department of Mechanical, Aerospace, and Nuclear Engineering. Member AIAA.

†Eppright Professor, Department of Aerospace Engineering. Fellow AIAA.

‡Graduate Student, Department of Aerospace Engineering. Student Member AIAA.

orthogonal, all its eigenvalues lie on the unit circle $S^1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$. Therefore, $sp(C) \subset S^1$, and the transformation in Eq. (2) requires that $-1 \notin sp(C)$. One can easily show that $C \in SO(n)$ if $Q \in so(n)$ (Ref. 7), and thus the Cayley transformation is injective (one-to-one) and surjective (onto) between the set of skew-symmetric matrices and the set of proper orthogonal matrices with no eigenvalue at -1.

III. Cayley Transforms as Conformal Mappings

The three most important subsets of the complex numbers are the real numbers $\mathbb{R}$, the imaginary numbers $\Im$, and the numbers with absolute value one (i.e., the numbers on the unit circle S^1). Trivially, these sets are subsets of the complex plane $\mathbb{C}$. There is a very elegant analog between these three subsets of the complex plane and the $n \times n$ matrices,[9] i.e., the elements of $\mathbb{R}^{n\times n}$. This analog can be easily understood and appreciated as follows: An elementary result in matrix algebra states that every $n \times n$ matrix with real elements can be decomposed into the sum of a symmetric and a skew-symmetric matrix. For example, any $A \in \mathbb{R}^{n\times n}$ can be written as $A = (A + A^T)/2 + (A - A^T)/2$. The first matrix in this equation is symmetric, and the second matrix is skew-symmetric. Symmetric matrices always have real eigenvalues, and skew-symmetric matrices always have imaginary eigenvalues. Recall now that a complex number can always be decomposed into the sum of a real and an imaginary part. This parallelism between complex numbers and matrices allows one to treat the symmetric matrices as the real numbers and the skew-symmetric matrices as the imaginary numbers in the set of $\mathbb{R}^{n\times n}$ matrices.[9] In addition, recall that an orthogonal matrix in $\mathbb{R}^{n\times n}$ has all of its eigenvalues on the unit circle. Drawing the previous parallelism even further, we can therefore treat the orthogonal matrices as the elements on the unit circle in the space $\mathbb{R}^{n\times n}$. Similar statements can be made for the case of $n \times n$ matrices with complex entries (elements of $\mathbb{C}^{n\times n}$), where now hermitian, skew-hermitian, and unitary matrices have to be used instead of symmetric, skew-symmetric, and orthogonal matrices, respectively.

We intend to use this heuristic correspondence between complex numbers and $n \times n$ matrices to motivate and generalize the Cayley transform to higher order. Before we proceed, we briefly review some elements from complex function theory.[11,12] First, recall that a (complex) function is analytic in an open set if it has a derivative at each point in that set. In particular, f is analytic at point z_0 if it is analytic in a neighborhood of z_0. Moreover, analytic functions have (uniformly) convergent power series expansions.[11]

A transformation $w = f(z)$, where $w, z \in \mathbb{C}$ is said to be conformal[11] at point z_0 if f is analytic there and $f'(z_0) \neq 0$. A conformal mapping is actually conformal at each point in a neighborhood of z_0, because the analyticity of f at z_0 implies analyticity in a neighborhood of z_0. Moreover, because f' is continuous at z_0, it follows that there is also a neighborhood of z_0 with $f'(z) \neq 0$ for all z in this neighborhood.[11] It is a trivial consequence of this definition that the composition of conformal mappings is also a conformal mapping.

A significant special class of conformal mappings in the complex plane is the class of linear fractional transformations (also called bilinear transformations) defined by

$$w = (az + b)/(cz + d) \qquad (ad - bc \neq 0) \tag{3}$$

An important property of the linear fractional transformations is that they always transform circles and lines into circles and lines.[11] In particular, in this paper we are interested in conformal transformations of the form in Eq. (3), which map the unit circle on the imaginary axis and vice versa. One such transformation is given by $w = f(z)$, where

$$f(z) = (1 - z)/(1 + z) \tag{4}$$

It is an easy exercise to show that if $z \in \Im$ then $|w| = 1$, that is, $w \in S^1$, and thus w is on the unit circle. Conversely, if $w \in S^1$ then the inverse transformation $z = f^{-1}(w)$ given by

$$f^{-1}(w) = (1 - w)/(1 + w) \tag{5}$$

implies that the real part of z is zero and thus, $z \in \Im$.

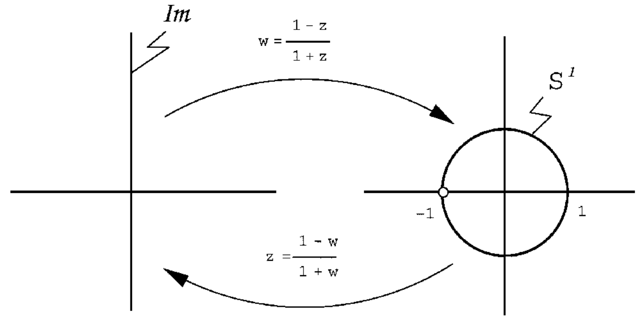

Fig. 1 Bilinear transformation.

The inverse transformation in Eq. (5) is defined everywhere except at $w = -1$. The point $w = -1$ is mapped to infinity (see Fig. 1). In fact, the map in Eq. (4) introduces a one-to-one transformation $f : \Im \to S^1 \backslash \{-1\}$.

Let us now introduce the conformal mapping $g_n : S^1 \to S^1$ defined by

$$g_n(w) = w^n \qquad n = 2, 3, \ldots \tag{6}$$

The function g_n is a mapping from the unit circle onto the unit circle. This transformation is injective only locally. Therefore, the inverse of g_n exists only locally. Given $\chi = e^{i\theta} \in S^1$, the solution of the equation $\chi = w^n$ ($n = 2, 3, \ldots$) yields that

$$w = \exp[i(\theta + 2k\pi)/n] \qquad k = 0, 1, 2, \ldots, n - 1 \tag{7}$$

Equation (7) shows that, in general, the equation $\chi = w^n$ has more than one solution. This result will turn out to be beneficial in Sec. V, where we discuss the application of higher-order Cayley transforms to attitude representations, because these roots can be used to avoid the inherent singularities of three-dimensional parameterizations of $SO(3)$. For $k = 0$ in Eq. (7), we get that $w = e^{i(\theta/n)}$. We will call this the principal nth root of χ.

The composition of the maps f and g_n is the function $h_n : \Im \to S^1$ defined by $h_n = g_n \circ f$, that is

$$h_n(z) = [(1 - z)/(1 + z)]^n \tag{8}$$

which maps the imaginary axis onto the unit circle. Similarly to g_n, this map is invertible only locally. A local inverse is obtained, for example, by setting $k = 0$ in Eq. (7), in which case we have that $z = [(1 - e^{i(\theta/n)})/(1 + e^{i(\theta/n)})] \in \Im$ [recall that $\chi = h_n(z) = e^{i\theta}$].

IV. Higher-Order Cayley Transforms

One of the most celebrated results in matrix algebra is the Cayley–Hamilton theorem. This theorem states that a matrix satisfies its own characteristic polynomial. An important consequence of this theorem is that, given any matrix $A \in \mathbb{R}^{n\times n}$ and an analytic function $F(z)$ inside a disk of radius r in the complex plane, one can unambiguously define the matrix-valued function $F(A)$ if the eigenvalues of A lie inside the disk of radius r. In other words, if F is given by

$$F(z) = \sum_{i=0}^{\infty} \alpha_i z^i (|z| \leq r)$$

then

$$F(A) = \sum_{i=0}^{\infty} \alpha_i A^i$$

and the previous series converges, assuming that $|\lambda_j| \leq r$, where $\lambda_j \in sp(A)$ for $j = 1, 2, \ldots, n$. Therefore, the matrix $F(A)$ is well-defined. Moreover, the eigenvalues of the matrix $F(A)$ are $F(\lambda_j)$ ($j = 1, 2, \ldots, n$) (Refs. 13 and 14).

Consider now the conformal mapping f from Eq. (4), which maps the imaginary axis on the unit circle. This function is analytic everywhere. According to the preceding discussion, the matrix

$$f(Q) = (I - Q)(I + Q)^{-1} = (I + Q)^{-1}(I - Q) \tag{9}$$

is well-defined for $Q \in so(n)$ and, actually, $C = f(Q) \in SO(n)$. Comparison between the previous equation and Eq. (1) reveals that the Cayley transform can be viewed as a special case of a conformal mapping in the space of matrices.

We have seen that there is a natural correspondence between $\Im$ and $so(n)$, as well as between S^1 and $SO(n)$. (We caution the mathematically inclined reader to take these statements in the context of the discussion in Sec. III. We do not claim that this correspondence carries any more weight than providing one qualitative motivation for the generalization of certain complex analytic results to analogous results in the space of matrices.) Following Eq. (8), we can also define a series of transformations $h_n : so(n) \to SO(n)$ by

$$h_n(Q) = (I - Q)^n (I + Q)^{-n} = (I + Q)^{-n} (I - Q)^n \qquad (10)$$

where Q is a skew-symmetric matrix. It should be clear by now that $C = h_n(Q)$ is a proper orthogonal matrix, i.e., $C \in SO(n)$. We shall refer to the family of maps $h_n(Q)$ in Eq. (10) as higher-order Cayley transforms. The consequences of such a generalization in attitude representations will become apparent in the next section.

For now, let us concentrate on the inverse map $h_n^{-1} : SO(n) \to so(n)$. Because $h_n = g_n \circ f$, one obtains $h_n^{-1} = f^{-1} \circ g_n^{-1}$. The function f^{-1} is given by Eq. (5), which, when applied to a proper orthogonal matrix Q with no eigenvalue at -1, gives the inverse of the classic (or first order) Cayley transform as in Eq. (2). The map $g_n^{-1} : SO(n) \to SO(n)$, on the other hand, requires the nth root of an orthogonal matrix. First, we show that g_n^{-1} is well-defined in the sense that the nth root of a (proper) orthogonal matrix with no eigenvalue at -1 is also a (proper) orthogonal matrix with no eigenvalue at -1. This will also prove that the composition of maps g_n^{-1} and f^{-1} is well-defined, because the range of g_n^{-1} is in the domain of f^{-1}.

To this end, consider an orthogonal matrix, $C \in SO(n)$, such that $\lambda \neq -1$ for all $\lambda \in sp(C)$. Because the matrix C is normal, it can be decomposed as $C = U \Theta U^*$ for some unitary matrix U, where $\Theta = \text{blockdiag}(\Theta_1, \Theta_2, \ldots, \Theta_{(n-1)/2}, +1)$ if n is odd, $\Theta = \text{blockdiag}(\Theta_1, \Theta_2, \ldots, \Theta_{n/2})$ if n is even, and $\Theta_j = \text{diag}(e^{i\theta_j}, e^{-i\theta_j})$. (A unitary matrix satisfies $UU^* = U^*U = I$, where U^* denotes the complex conjugate transpose of the matrix U.) The diagonal elements of the matrix Θ are the eigenvalues of C. The principal kth root of the matrix C is then given by $W = U \Theta^{1/k} U^*$, where $W^k = C$ and $\Theta^{1/k} = \text{blockdiag}(\Theta_1^{1/k}, \Theta_2^{1/k}, \ldots, \Theta_{(n-1)/2}^{1/k}, +1)$ if n is odd, $\Theta^{1/k} = \text{blockdiag}(\Theta_1^{1/k}, \Theta_2^{1/k}, \ldots, \Theta_{n/2}^{1/k})$ if n is even, and $\Theta_j^{1/k} = \text{diag}(e^{i(\theta_j/k)}, e^{-i(\theta_j/k)})$. Because $e^{i\theta_j} \neq -1$ for all $j = 1, \ldots, n\ (n-1)$, the angles $\theta_j \neq \pm 180$ deg and thus also $\theta_j / k \neq \pm 180$ deg for $k = 2, 3, \ldots,$ and thus $e^{i(\theta_j/k)} \neq -1$. Note that, to keep W proper, we always choose the positive root of the eigenvalue $+1$.

V. Attitude Representations

In this section, we concentrate on the ramifications of the previously developed results to attitude representations. Our motivation for investigating Cayley transforms in the first place stems from the fact that proper orthogonal matrices represent rotations. In particular, $SO(3)$ is the configuration space of all three-dimensional rotations. In other words, every element of $SO(3)$ represents a physical rotation between two reference frames in $\mathbb{R}^3$ and, conversely, every rotation can be represented by an element in $SO(3)$.

As a reference frame, namely, a body, rotates freely in the three-dimensional space, the corresponding rotation matrix C traces a curve in $SO(3)$ so that $C(t) \in SO(3)$ for all $t \geq 0$. The differential equation characterizing this trajectory on $SO(3)$ is given by

$$\dot{C} = [\omega] C \qquad (11)$$

where, given a vector $\omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3$, the matrix $[\omega]$ is defined by

$$[\omega] = \begin{bmatrix} 0 & \omega_3 & -\omega_2 \\ -\omega_3 & 0 & \omega_1 \\ \omega_2 & -\omega_1 & 0 \end{bmatrix} \qquad (12)$$

In the sequel, we apply the results of the previous section to parameterize the rotation group. In particular, the series of conformal mappings from Eq. (10) provides a family of parameters on $SO(3)$. Before undertaking this task, we investigate another important conformal mapping.

A. Exponential Map and Euler Parameters

Linear fractional transformations are not the only class of conformal mappings from the imaginary axis onto the unit circle. The exponential map, defined by

$$w = \exp(z) = e^z \qquad (13)$$

also maps $\Im$ (actually the strip $-i\pi \leq z \leq i\pi$) onto S^1. Clearly, if $z = i\theta$ then $|w| = 1$. The inverse transformation is

$$z = \log w = i(\theta + 2n\pi) \qquad n = 0, \pm 1, \pm 2, \ldots \qquad (14)$$

and is defined only locally.

We can therefore define the exponential map from the space of skew-symmetric matrices to the space of proper orthogonal matrices. This exponential map is defined, as usual, by

$$C = e^Q = \sum_{n=0}^{\infty} \frac{1}{n!} Q^n \qquad (15)$$

and the series converges for every Q. One can easily show that C thus defined is indeed proper orthogonal. For the three-dimensional case, the matrix $Q \in so(3)$ can be parameterized by $Q = [\beta]$. As before, given a vector $\beta = (\beta_1, \beta_2, \beta_3)^T \in \mathbb{R}^3$, we use the notation $[\beta]$ to denote the skew-symmetric matrix in Eq. (12). Euler's formula[4] yields

$$e^{[\beta]} = I + \sin\chi([\beta]/\chi) + (1 - \cos\chi)([\beta]^2/\chi^2) \qquad (16)$$

where $\chi = \|\beta\|$. Normalizing the vector β, we get a unit vector

$$\hat{e} = \beta / \|\beta\| \qquad (17)$$

Euler's theorem[1] states that any rotation can be represented by a finite rotation (principal rotation) about a single axis (principal axis). That is, the principal axis and the principal angle suffice to determine the rotation matrix. From a mathematical perspective, this amounts to parameterizing elements in $SO(3)$ by the principal axis and the principal angle.

By letting the principal axis be along the direction of the unit vector $\hat{e}$ and by letting the principal angle be χ as earlier, Eq. (16) shows how this parameterization is achieved. Clearly,

$$C(\chi, \hat{e}) = e^{\chi[\hat{e}]} \qquad (18)$$

Moreover, introducing the Euler parameter vector $q = (q_0, q_1, q_2, q_3)^T$

$$q_0 = \cos(\chi/2) \qquad q_i = \hat{e}_i \sin(\chi/2) \qquad i = 1, 2, 3 \qquad (19)$$

and substituting in Eq. (16), one obtains the well-known formula for the rotation matrix in terms of the Euler parameters[4]:

$$C(q) = \left(q_0^2 - \tilde{q}^T \tilde{q}\right) I + 2\tilde{q}\tilde{q}^T + 2q_0[\tilde{q}] \qquad (20)$$

where $\tilde{q} = (q_1, q_2, q_3)^T \in \mathbb{R}^3$ is the vector part of the Euler parameters.

Therefore, the Euler parameter representation as well as the Euler axis/angle representation are obtained by generalizing the conformal mapping in Eq. (13) to the space of matrices. Note from Eq. (20) that $C(q) = C(-q)$ and both q and $-q$ can be used to describe the same physical orientation. This fact can be used to construct alternative, or shadow, sets of kinematic parameters obtained via the Cayley transforms.

B. Rodrigues Parameters

Because the Euler parameters satisfy the additional constraint $q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1$, one is naturally led to consider elimination of this constraint, thus reducing the number of coordinates from four to three. The Rodrigues parameters achieve this by defining

$$\rho_j = q_j / q_0 \qquad j = 1, 2, 3 \tag{21}$$

The three parameters ρ_1, ρ_2, ρ_3 then provide a three-dimensional parameterization of $SO(3)$. The inverse transformation of Eq. (21) is given by

$$q_0 = \frac{1}{(1+\hat{\rho}^2)^{\frac{1}{2}}} \qquad q_j = \frac{\rho_j}{(1+\hat{\rho}^2)^{\frac{1}{2}}} \qquad j = 1, 2, 3 \tag{22}$$

where $\hat{\rho}^2 = \rho^T\rho = \rho_1^2 + \rho_2^2 + \rho_3^2$. The Rodrigues parameters are related to the principal axis and angle through the equation

$$\rho = \tan(\chi/2)\,\hat{e} \tag{23}$$

The rotation matrix in terms of the Rodrigues parameters can be easily computed by Eqs. (20) and (22):

$$C(\rho) = \frac{1}{1+\hat{\rho}^2}\left\{(1-\hat{\rho}^2)I + 2\rho\rho^T + 2[\rho]\right\} \tag{24}$$

It is remarkable that the previous parameterization of $SO(3)$ can also be achieved by means of the Cayley transformation in Eq. (1). If we introduce the skew-symmetric matrix $R = -[\rho]$, the transformation

$$C = (I-R)(I+R)^{-1} = (I+R)^{-1}(I-R) \tag{25}$$

produces exactly the matrix in Eq. (24). Therefore, the classic Cayley–Rodrigues parameters representation is obtained by generalizing the conformal mapping in Eq. (4) to the space of matrices.

C. Modified Rodrigues Parameters

The normalization in Eq. (21) is not the only possible one. A more judicious normalization for eliminating the Euler parameter constraint is through stereographic projection.[11,12,15,16] By this approach, the new variables

$$\sigma_j := q_j/(1+q_0) \qquad j = 1, 2, 3 \tag{26}$$

provide another set of parameters on $SO(3)$. These parameters are referred to in the literature as the modified Rodrigues parameters[4] and have distinct advantages over the classic Rodrigues parameters. In particular, whereas the Rodrigues parameters do not allow eigenaxis rotations of more than 180 deg, the modified Rodrigues parameters allow for eigenaxis rotations of up to 360 deg.[6,7,15–17] This can be deduced immediately by the corresponding relationship between σ and the principal axis and angle

$$\sigma = \tan(\chi/4)\,\hat{e} \tag{27}$$

which is well-behaved for $0 \le \chi < 2\pi$. Because both q and $-q$ describe the same physical orientation (recall the discussion at the end of Sec. V.A), a second set of parameters defined by

$$\sigma_j^s := -q_j/(1-q_0) \qquad j = 1, 2, 3 \tag{28}$$

referred to as the shadow set[16] can be used to describe the same physical orientation. These parameters are also given by

$$\sigma^s = -\frac{1}{\tan(\chi/4)}\,\hat{e} \tag{29}$$

The transformation between σ and σ^s is given by[16]

$$\sigma^s = -\sigma/\hat{\sigma}^2 \tag{30}$$

where $\hat{\sigma}^2 = \sigma^T\sigma = \sigma_1^2 + \sigma_2^2 + \sigma_3^2 = \tan^2(\chi/4)$. The rotation matrix associated with the modified Rodrigues parameters is given by[4,7,15]

$$C(\sigma) = I + 4\frac{1-\hat{\sigma}^2}{(1+\hat{\sigma}^2)^2}[\sigma] + \frac{8}{(1+\hat{\sigma}^2)^2}[\sigma]^2 \tag{31}$$

In Ref. 7, it was shown that these parameters can also be defined by a Cayley transformation of second order. That is, if $S = -[\sigma]$ then the transformation

$$C = (I-S)^2(I+S)^{-2} = (I+S)^{-2}(I-S)^2 \tag{32}$$

produces exactly the matrix in Eq. (31). Note that the inverse of the transformation (32) is not unique, and it requires the square root of an orthogonal matrix. Given $C \in SO(3)$, we need to find matrix W such that $C = W^2$. Once matrix W is calculated, the skew-symmetric matrix S containing the modified Rodrigues parameters is computed from

$$S = (I-W)(I+W)^{-1} = (I+W)^{-1}(I-W) \tag{33}$$

Reference 7 outlines this approach. To every orthogonal matrix corresponds a principal angle and a principal direction according to Eq. (18). From Eqs. (18) and $C = W^2$, one therefore has that

$$W = e^{(\chi/2)[\hat{e}]} \tag{34}$$

and W has half the principal angle of C. It should be apparent now how the modified Rodrigues parameters double the domain of validity of the parameterization by taking the square of the classic Cayley transform.

This observation motivates the search of higher-dimensional Cayley transforms for attitude representations. Such transformations are expected to increase the domain of validity even further. This is the topic of the next section.

D. Higher-Order Rodrigues Parameters

According to the discussion in the preceding section, one expects that higher-order Cayley transformations will increase the domain of validity of the corresponding parameters. The main task of this section is to derive these higher-order parameters and find their connections to the Rodrigues parameters, the modified parameters, and the Euler parameters. To this end, consider first the fourth-order Cayley transform defined by

$$C = (I-T)^4(I+T)^{-4} \tag{35}$$

for some skew-symmetric matrix $T = -[\upsilon]$. We know that the matrix C is (proper) orthogonal.

Let $\upsilon = (\upsilon_1, \upsilon_2, \upsilon_3)^T \in \mathbb{R}^3$ be the vector of these parameters. Our purpose is to establish connections between the attitude parameters υ and the other classic attitude parameters such as the Euler parameters of the modified Rodrigues parameters.

Recall from the results of Sec. III that, if F is analytic function, then the eigenvalues of the matrix $F(A)$ are given by $F(\lambda_j)$, where λ_j are the eigenvalues of A. It is an easy exercise to show that the eigenvalues of the skew-symmetric matrix T are given by $\{0, \pm i\,(\upsilon_1^2+\upsilon_2^2+\upsilon_3^2)^{1/2}\}$. Similarly, the eigenvalues of the matrix S are given by $\{0, \pm i\,(\sigma_1^2+\sigma_2^2+\sigma_3^2)^{1/2}\}$. Let λ_υ denote an eigenvalue of T and λ_σ an eigenvalue of S. Comparing Eqs. (32) and (35), one sees that the matrices S and T are related by

$$(I-S)(I+S)^{-1} = (I-T)^2(I+T)^{-2} \tag{36}$$

This suggests that λ_σ and λ_υ are related by

$$(1-\lambda_\sigma)/(1+\lambda_\sigma) = [(1-\lambda_\upsilon)/(1+\lambda_\upsilon)]^2 \tag{37}$$

or

$$1+\lambda_\sigma = \frac{(1+\lambda_\upsilon)^2}{1+\lambda_\upsilon^2} \tag{38}$$

Solving for λ_σ and substituting the expressions for λ_σ and λ_υ in the preceding equation, one obtains that

$$\pm i\left(\sigma_1^2+\sigma_2^2+\sigma_3^2\right)^{\frac{1}{2}} = \pm 2i\,\frac{\left(\upsilon_1^2+\upsilon_2^2+\upsilon_3^2\right)^{\frac{1}{2}}}{1-\upsilon_1^2-\upsilon_2^2-\upsilon_3^2} \tag{39}$$

Upon squaring this expression, one obtains

$$\sigma_1^2 + \sigma_2^2 + \sigma_3^2 = 4\frac{\upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2}{\left(1 - \upsilon_1^2 - \upsilon_2^2 - \upsilon_3^2\right)^2} \tag{40}$$

This equation suggests that σ and υ are related by

$$\sigma_j = \pm\frac{2\upsilon_j}{1 - \hat{\upsilon}^2} \qquad j = 1, 2, 3 \tag{41}$$

where $\hat{\upsilon}^2 = \upsilon_1^2 + \upsilon_2^2 + \upsilon_3^2$. Arbitrarily, and without loss of generality, we choose the solution with the plus sign. Substitution in S and computing C from Eq. (32) verifies the expression in Eq. (41).

The relation between υ and q is obtained by observing that

$$\frac{2\upsilon_j}{1 - \hat{\upsilon}^2} = \frac{q_j}{1 + q_0} \qquad j = 1, 2, 3 \tag{42}$$

After some calculations, one obtains that

$$\frac{2}{1 - \hat{\upsilon}^2} = \frac{\pm\sqrt{2} + \sqrt{1 + q_0}}{\sqrt{1 + q_0}} \tag{43}$$

Using Eq. (42), one now obtains that

$$\upsilon_j = \frac{q_j}{1 + q_0 \pm \sqrt{2(1 + q_0)}} \qquad j = 1, 2, 3 \tag{44}$$

Conversely, starting from

$$1 + q_0 = 2\left(\frac{1 - \hat{\upsilon}^2}{1 + \hat{\upsilon}^2}\right)^2 \tag{45}$$

and using Eq. (42), one obtains that the Euler parameters are given in terms of the υ parameters from

$$q_j = 4\upsilon_j\frac{1 - \hat{\upsilon}^2}{(1 + \hat{\upsilon}^2)^2} \qquad j = 1, 2, 3 \tag{46}$$

and

$$q_0 = 2\left(\frac{1 - \hat{\upsilon}^2}{1 + \hat{\upsilon}^2}\right)^2 - 1 = \frac{1 - 6\hat{\upsilon}^2 + \hat{\upsilon}^4}{(1 + \hat{\upsilon}^2)^2} \tag{47}$$

where $\hat{\upsilon}^4 = (\hat{\upsilon}^2)^2$. Letting $W = (I - T)(I + T)^{-1}$ and because $C = W^4$, one obtains that

$$W = e^{(\chi/4)[\tilde{e}]} \tag{48}$$

where χ is the principal angle of C. Moreover, using the definition of the Euler parameters from Eq. (19), one obtains the following result for the υ parameters:

$$\upsilon = \frac{\sin(\chi/2)}{1 + \cos(\chi/2) \pm 2\cos(\chi/4)}\,\hat{e} \tag{49}$$

where $\hat{e}$ is the unit vector along the principal axis. Keeping the plus sign, Eq. (49) can be further reduced to the simple formula

$$\upsilon_+ = \tan(\chi/8)\,\hat{e} \qquad (-4\pi < \chi < 4\pi) \tag{50}$$

From Eq. (50), it is apparent that υ is proportional to the principal rotation axis, like the classic and modified Rodrigues parameters, where the proportionality factor is now $f(\chi) = \tan(\chi/8)$. Equation (50) is reassuring because it proves that the υ parameters indeed behave as higher-order Rodrigues parameters, which can be used to linearize the domain of validity of the kinematic parameterization. By this, we mean that Eq. (50) behaves almost linearly as a function of the principal angle χ (especially in the region $-\pi \le \chi \le \pi$); see also Fig. 2.

If we choose the minus sign in Eq. (49), we obtain that

$$\upsilon_- = -\frac{1}{\tan(\chi/8)}\,\hat{e} \qquad (0 < \chi < 8\pi) \tag{51}$$

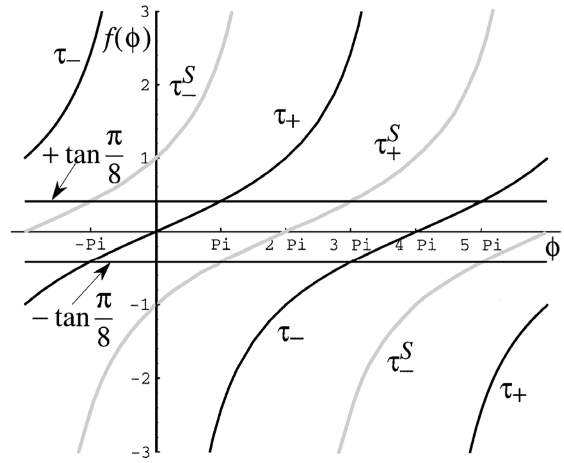

Fig. 2 Comparison of original and shadow τ parameters.

Moreover, reversing the signs of the Euler parameters in Eq. (44), one obtains that the υ parameters have a unique set of shadow parameters like the modified Rodrigues parameters.[16] These parameters are obtained by setting

$$\upsilon^s = \frac{-\sin(\chi/2)}{1 - \cos(\chi/2) \pm 2\sin(\chi/4)}\,\hat{e} \tag{52}$$

It can be easily verified that the corresponding shadow parameters reduce to

$$\upsilon_+^s = \frac{\tan(\chi/8) - 1}{\tan(\chi/8) + 1}\,\hat{e} \qquad (-2\pi < \chi < 6\pi) \tag{53}$$

and

$$\upsilon_-^s = \frac{1 + \tan(\chi/8)}{1 - \tan(\chi/8)}\,\hat{e} \qquad (-6\pi < \chi < 2\pi) \tag{54}$$

As the original υ parameters approach $+1$, the associated shadow parameters υ^s approach zero and vice versa. The general transformation between the original and the shadow set is given by

$$\upsilon^s = -\upsilon\left[\frac{1 - \hat{\upsilon}^2}{2\hat{\upsilon}^2 + (1 + \hat{\upsilon}^2)\hat{\upsilon}}\right] \tag{55}$$

where $\hat{\upsilon} = (\hat{\upsilon}^2)^{1/2}$. Equations (50), (51), (53), and (54) can be used to compute the four distinct roots of Eq. (35). Note also that Eqs. (50), (53), (51), and (54) can be also written in the form

$$\upsilon = \tan[(\chi/8) - k(\pi/4)]\,\hat{e} \qquad k = 0, 1, 2, 3 \tag{56}$$

respectively. The shadow parameter set υ^s is shown side by side with the original υ parameters in Fig. 2. The shadow set is plotted in gray. Figure 2 also shows that υ parameters are indeed very linear for small rotations within ± 180 deg.

As with the modified Rodrigues parameters (and other stereographic parameters),[16] these shadow parameters represent the same physical orientation as the original set and abide by the same differential kinematic equation. They could be used to avoid the problems of approaching the ± 720-deg principal rotation. By switching to the shadow trajectory, all numerical problems would be avoided. Having, however, a principal rotation range of ± 720 deg is really more than needed. Limiting the principal rotations to be within ± 180 deg would suffice and be much more attractive. As the magnitude of υ approaches $\tan(\pi/8)$, then one would simply switch the υ to their shadow set. Having $\|\upsilon\| = \tan(\pi/8)$ corresponds to $q_0 = 0$. From Eq. (44), one can then see that, at this point, the two sets of parameters are related by $\upsilon = -\upsilon^s$. The combined set of original and shadow υ parameters would provide a set of attitude coordinates that are very linear with respect to the principal rotation angle, more so even than the modified Rodrigues parameters. We note in passing that the previous approach can be easily extended to any Cayley transform of order 2^k, because Eqs. (36) and (37) can be used iteratively.

For the third-order Cayley transform, we have that

$$C = (I - P)^3(I + P)^{-3} = (I + P)^{-3}(I - P)^3 \quad (57)$$

where $P = -[p]$ and $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$ the corresponding parameters. If λ_ρ and λ_p denote the respective eigenvalues of the skew-symmetric matrices R and P then, using Eqs. (25) and (57), they must be related by

$$(1 - \lambda_\rho)/(1 + \lambda_\rho) = [(1 - \lambda_p)/(1 + \lambda_p)]^3 \quad (58)$$

Upon expanding the preceding equality and solving for λ_ρ, one obtains

$$\lambda_\rho = \frac{\lambda_p\left(3 + \lambda_p^2\right)}{1 + 3\lambda_p^2} \quad (59)$$

The preceding equation suggests that ρ_j and p_j are related by

$$\rho_j = \pm\frac{p_j\left(3 - p_1^2 - p_2^2 - p_3^2\right)}{1 - 3\left(p_1^2 + p_2^2 + p_3^2\right)} \qquad j = 1, 2, 3 \quad (60)$$

To get the relation of p to the Euler parameter vector, one can set

$$\frac{p_j\left(3 - p_1^2 - p_2^2 - p_3^2\right)}{1 - 3\left(p_1^2 + p_2^2 + p_3^2\right)} = \frac{q_j}{q_0} \quad (61)$$

and solve for $\hat{p}^2 = p_1^2 + p_2^2 + p_3^2$. After some algebraic calculations, it is not difficult to show that, in fact,

$$\frac{(\hat{p}^2 + 1)^3}{(1 - 3\hat{p}^2)^2} = \frac{1}{q_0^2} \quad (62)$$

Solution of the preceding equation for $\hat{p}^2$ requires the solution of a cubic equation. Once $\hat{p}^2$ is known, however, it can be substituted into Eq. (61) to get the desired result. Actually, from Eqs. (61) and (62) we have that

$$q_0 = \frac{1 - 3\hat{p}^2}{(1 + \hat{p}^2)^{\frac{3}{2}}} \qquad q_j = \pm\frac{p_j(3 - \hat{p}^2)}{(1 + \hat{p}^2)^{\frac{3}{2}}} \qquad j = 1, 2, 3 \quad (63)$$

Letting $W = (I - P)(I + P)^{-1}$, then because $C = W^3$ one obtains that

$$W = e^{(\chi/3)[\hat{e}]} \quad (64)$$

where χ is the principal angle of C. A straightforward but tedious calculation shows that the parameters p are related to the principal axis and angle through

$$p = \tan(\chi/6)\,\hat{e} \qquad (-3\pi < \chi < 3\pi) \quad (65)$$

Similarly to Eqs. (50) to (54), multiple solutions with the shadow set can also be derived and are left to the interested reader.

VI. Kinematics

The kinematic equations in terms of the υ parameters can be computed as follows. From Eqs. (11) and (35) we have that

$$\dot{C} = \frac{\mathrm{d}}{\mathrm{d}t}[(I - T)^4](I + T)^{-4} + (I - T)^4\frac{\mathrm{d}}{\mathrm{d}t}[(I + T)^{-4}]$$
$$= [\omega](I - T)^4(I + T)^{-4} \quad (66)$$

or that

$$\frac{\mathrm{d}}{\mathrm{d}t}[(I - T)^4] - C(T)\frac{\mathrm{d}}{\mathrm{d}t}[(I + T)^4] = [\omega](I - T)^4 \quad (67)$$

where we have used the fact that $\mathrm{d}A^{-1}/\mathrm{d}t = -A^{-1}(\mathrm{d}A/\mathrm{d}t)A^{-1}$ for any square matrix A. Using also the fact that

$$\frac{\mathrm{d}A^n}{\mathrm{d}t} = \sum_{j=0}^{n-1} A^j\frac{\mathrm{d}A}{\mathrm{d}t}A^{n-j-1}$$

and performing the differentiations in the left-hand side of Eq. (67), one obtains a set of nine linear equations in terms of $\dot{\upsilon}_1$, $\dot{\upsilon}_2$, and $\dot{\upsilon}_3$. Similarly, the right-hand side of Eq. (67) is linear in terms of ω_1, ω_2, and ω_3. Choosing three (independent) equations out of these nine, we get a linear system of the form

$$V(\upsilon)\,\dot{\upsilon} = U(\upsilon)\,\omega \quad (68)$$

Solving for $\dot{\upsilon}$, we finally get that the kinematic equations for the υ orientation parameters are given by

$$\frac{\mathrm{d}\upsilon}{\mathrm{d}t} = V^{-1}(\upsilon)\,U(\upsilon)\,\omega = G(\upsilon)\omega \quad (69)$$

where the matrix $G(\upsilon)$ is given by

$$G(\upsilon) = \frac{1}{8(1 - \hat{\upsilon}^2)}\{2(3 - \hat{\upsilon}^2)\upsilon\upsilon^T - 4(1 - \hat{\upsilon}^2)[\upsilon]$$
$$+ (1 - 6\hat{\upsilon}^2 + \hat{\upsilon}^4)I\} \quad (70)$$

These kinematic equations are not as simple as the corresponding kinematic equations for the Rodrigues or the modified Rodrigues parameters.[7,15,16] The limiting behavior of these equations as $\hat{\upsilon} \to \pm 1$ will be investigated next. We will show that Eq. (69) is actually well-defined and the apparent singularity at $\hat{\upsilon} = \pm 1$, equivalently at $\chi = \pm 2\pi$, is removable.

To this end, denote by $\|\cdot\|_2$ the Frobenius norm of real matrix A, i.e., $\|A\|_2^2 = \mathrm{tr}(A^T A)$. For the kinematic equations in Eq. (69) then, after some laborious but straightforward calculations, one obtains

$$\|G(\upsilon)\|_2^2 = \mathrm{tr}[G(\upsilon)^T G(\upsilon)]$$
$$= \frac{1}{64(1 - \hat{\upsilon}^2)}\,\mathrm{tr}\{4(3 - \hat{\upsilon}^2)^2\hat{\upsilon}^2\upsilon\upsilon^T$$
$$+ 4(3 - \hat{\upsilon}^2)(1 - 6\hat{\upsilon}^2 + \hat{\upsilon}^4)\upsilon\upsilon^T$$
$$+ (1 - 6\hat{\upsilon}^2 + \hat{\upsilon}^4)^2 I - 16(1 - \hat{\upsilon}^2)^2[\upsilon]^2\} \quad (71)$$

From the definition of the matrix $[\upsilon]$, we have that $[\upsilon]^2 = \upsilon\upsilon^T - \hat{\upsilon}^2 I$. Substituting in Eq. (71) and noting that $\mathrm{tr}(\upsilon\upsilon^T) = \hat{\upsilon}^2$, Eq. (71) reduces to

$$\|G(\upsilon)\|_2^2 = \frac{(1 - \hat{\upsilon}^4)^2}{64(1 - \hat{\upsilon}^2)^2} = \frac{(1 + \hat{\upsilon}^2)^2}{64} \quad (72)$$

thus,

$$\lim_{\hat{\upsilon}^2 \to 1}\|G(\upsilon)\|_2 = \tfrac{1}{4} < \infty \quad (73)$$

The last equation implies that the behavior of the υ parameters is well-conditioned at $\chi = \pm 2\pi$. In addition, because of the near-linear behavior between χ and the magnitude of υ, for small principal angles, Eq. (69) is expected to behave in a more linear-like fashion than either the Cayley–Rodrigues or the modified Rodrigues parameters.

Similarly, for the third-order Cayley parameters, one can derive the following kinematic equations:

$$\frac{\mathrm{d}p}{\mathrm{d}t} = \frac{1}{6(3 - \hat{p}^2)}\{(11 - \hat{p}^2)pp^T - 3(3 - \hat{p}^2)[p]$$
$$+ 3(1 - 3\hat{p}^2)I\}\omega \quad (74)$$

These equations can be derived starting from Eqs. (11) and (57) and using similar arguments as before. A similar analysis shows that the limiting behavior of this system as $\hat{p} \to \pm\sqrt{3}$ is well-defined and no singularity is encountered during integration. This is also verified through numerical simulations in the next section. In fact, for all of the parameters σ, p, and υ, the singularity of the kinematics is entirely due to the same mechanism, as for the classic Cayley–Rodrigues parameters ρ. Namely, the kinematic differential equations themselves are well-defined and continuous functions (as opposed to the Eulerian angle case) but the quadratic and higher-order polynomial nonlinearities induce the possibility of finite escape times.

VII. Numerical Example

To demonstrate the potential benefits of the kinematic parameters, we present the results of the following simulations. We integrated Eqs. (69) and (74) as well as the corresponding kinematic equations in terms of the Cayley–Rodrigues (ρ) and the modified Rodrigues parameters (σ) starting from the zero orientation and subject to the constant angular velocity vector $\omega = (0.25, 0.4, -0.1)$(rad/s). This corresponds to a linearly increasing value of the principal angle χ. The results of the simulations are shown in Fig. 3. This figure actually shows only the first components of the kinematic parameter vectors, as the other two components exhibit similar behavior.

As is evident, the classic and the modified Rodrigues parameters encounter the singularity earlier than the υ and p parameters. Also, the p parameters become singular earlier than the υ parameters. We note, however, that because discontinuities in the parameter description are typically acceptable in applications, the modified Rodrigues parameters can be made to avoid the singularity altogether by simply switching to their shadow set. The same also holds for the υ parameters via Eq. (55) or the p parameters. Figure 4 shows the simulation where the parameters σ and υ are allowed to switch to their respective shadow sets. Although the points of switching are arbitrary and can be chosen according to the particular application, a reasonable choice is to switch when the parameters and the corresponding shadow set have opposite signs. This ensures continuity of the magnitude. From Eqs. (30) and (55), this occurs when $\chi = k\pi$, $k = \pm 1, \pm 2, \ldots$. This is the situation depicted in Fig. 4. The υ parameters are shown as solid lines, and the σ parameters are shown as dashed lines. Because the classic Rodrigues parameters do not have an associated shadow set (better, the shadow set coincides with the original parameters), only the σ and υ parameters are plotted in Fig. 4.

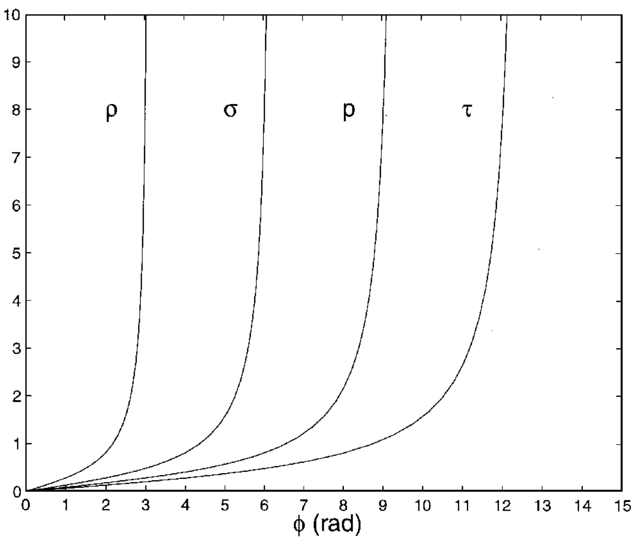

Fig. 3 Orientation parameter comparison.

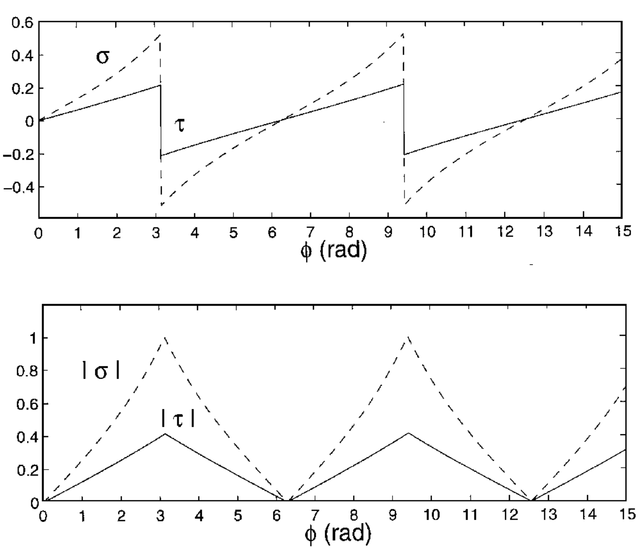

Fig. 4 Orientation parameters and their shadow sets.

VIII. Conclusions

We have extended the classic Cayley transform, which maps skew-symmetric matrices to proper orthogonal matrices to higher orders. The approach is based on the observation that Cayley transforms can be viewed as generalized conformal (bilinear) mappings in the space of matrices. The Euler parameters, the Rodrigues parameters, and the modified Rodrigues parameters follow as special cases of this approach. In addition, we have generated a family of higher-order Rodrigues parameters, which could be used as parameters for the rotation group. It still remains, however, to determine the applicability of these higher order parameters in realistic attitude problems.

Acknowledgments

The work of the first author was supported in part by the National Science Foundation under Career Grant CMS-9624188 and in part by the University of Virginia SEAS Dean's Fund.

References

[1]Hughes, P. C., *Spacecraft Attitude Dynamics*, Wiley, New York, 1986, pp. 15–31.

[2]Markley, F. L., "Parameterizations of the Attitude," *Spacecraft Attitude Determination and Control*, edited by J. R. Wertz, Reidel, Dordrecht, The Netherlands, 1978, pp. 410–420.

[3]Stuelpnagel, J., "On the Parameterization of the Three-Dimensional Rotation Group," *SIAM Review*, Vol. 6, No. 4, 1964, pp. 422–430.

[4]Shuster, M. D., "A Survey of Attitude Representations," *Journal of the Astronautical Sciences*, Vol. 41, No. 4, 1993, pp. 439–517.

[5]Wiener, T. F., "Theoretical Analysis of Gimballess Inertial Reference Equipment Using Delta-Modulated Instruments," Ph.D. Thesis, Dept. of Aeronautical and Astronautical Engineering, Massachusetts Inst. of Technology, Cambridge, MA, March 1962.

[6]Marandi, S. R., and Modi, V., "A Preferred Coordinate System and the Associated Orientation Representation in Attitude Dynamics," *Acta Astronautica*, Vol. 15, No. 11, 1987, pp. 833–843.

[7]Schaub, H., Tsiotras, P., and Junkins, J. L., "Principal Rotation Representations of Proper $N \times N$ Orthogonal Matrices," *International Journal of Engineering Science*, Vol. 33, No. 15, 1995, pp. 2277–2295.

[8]Junkins, J. L., and Kim, Y., *Introduction to Dynamics and Control of Flexible Structures*, AIAA Education Series, AIAA, Washington, DC, 1993, pp. 53–57.

[9]Halmos, P. R., *Finite Dimensional Vector Spaces*, Vol. 7, Annals of Mathematics Studies, Princeton Univ. Press, Princeton, NJ, 1953, pp. 105–118.

[10]Curtis, M. L., *Matrix Groups*, Springer–Verlag, New York, 1979, pp. 25–43.

[11]Churchill, R. V., and Brown, J. W., *Complex Variables and Applications*, McGraw–Hill, New York, 1990.

[12]Conway, J. B., *Functions of One Complex Variable*, Springer–Verlag, New York, 1978.

[13]Horn, R., and Johnson, C. R., *Matrix Analysis*, Cambridge Univ. Press, Cambridge, England, UK, 1985, p. 300.

[14]Brogan, W. L., *Modern Control Theory*, 3rd ed., Prentice–Hall, Englewood Cliffs, NJ, 1991, p. 286.

[15]Tsiotras, P., "On New Parameterizations of the Rotation Group in Attitude Kinematics," TR, Dept. of Aeronautics and Astronautics, Purdue Univ., West Lafayette, IN, Jan. 1994.

[16]Schaub, H., and Junkins, J. L., "Stereographic Orientation Parameters for Attitude Dynamics: A Generalization of the Rodrigues Parameters," *Journal of the Astronautical Sciences*, Vol. 44, No. 1, 1996, pp. 1–19.

[17]Tsiotras, P., "New Control Laws for the Attitude Stabilization of Rigid Bodies," *Proceedings of the Thirteenth IFAC Symposium on Automatic Control in Aerospace* (Palo Alto, CA), 1994, pp. 316–321.